

# Large deviation exponential inequalities for supermartingales

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## Abstract

Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of supermartingale differences and let  $S_k = \sum_{i=1}^k X_i$ . We give an exponential moment condition under which  $P(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^\alpha\})$ ,  $n \rightarrow \infty$ , where  $\alpha \in (0, 1)$  is given and  $C_1 > 0$  is a constant. We also show that the power  $\alpha$  is optimal under the given condition. In particular, when  $\alpha = \frac{1}{3}$ , we recover an inequality of Lesigne and Volný.

*Keywords:* Large deviations; martingales; sub-exponential distributions

*2000 MSC:* 60F10, 60G42

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## 1. Introduction

Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of martingale differences and let  $S_k = \sum_{i=1}^k X_i$ ,  $k \geq 1$ . Under the Cramér condition  $\sup_i Ee^{|X_i|} < \infty$ , Lesigne and Volný [9] proved that

$$P(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \rightarrow \infty, \quad (1)$$

for some constant  $C_1 > 0$ . Here and throughout the paper, for two functions  $f$  and  $g$ , we write  $f(n) = O(g(n))$  if there exists a constant  $C > 0$  such that  $|f(n)| \leq C|g(n)|$  for all  $n \geq 1$ . Lesigne and Volný [9] also showed that the power  $\frac{1}{3}$  in (1) is optimal in the sense that there exists a sequence of martingale differences  $(\hat{X}_i, \mathcal{F}_i)_{i \geq 1}$  such that  $\sup_i Ee^{|\hat{X}_i|} < \infty$  and  $P(\hat{S}_n \geq n) > \exp\{-C_2 n^{\frac{1}{3}}\}$  for some constant  $C_2 > 0$  and infinitely many  $n$ 's. Liu and Watbled [10] proved that the power  $\frac{1}{3}$  in (1) can be improved to 1 under the conditional Cramér condition  $\sup_i E(e^{|X_i|} | \mathcal{F}_{i-1}) \leq C_3$ , for some constant  $C_3$ . It seems natural to ask under what condition it holds

$$P(S_n \geq n) = O(\exp\{-C_1 n^\alpha\}), \quad n \rightarrow \infty, \quad (2)$$

where  $\alpha \in (0, 1)$  is given and  $C_1 > 0$  is a constant. In this paper, we give some sufficient conditions in order that (2) holds for supermartingales  $(S_k, \mathcal{F}_k)_{k \geq 1}$ .

The paper is organized as follows. In Section 2, we present the main results. In Sections 3-5, we give the proofs of the main results.

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## 2. Main Results

Our first result is an extension of the bound (1) of Lesigne and Volný.

**Theorem 2.1.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $\sup_i E \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$  for some constant  $C_1 \in (0, \infty)$ . Then, for all  $x > 0$ ,*

$$P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq C(\alpha, x) \exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^\alpha\right\},$$

where

$$C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)$$

does not depend on  $n$ . In particular, with  $x = 1$ , it holds

$$P\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O\left(\exp\left\{-\frac{1}{16} n^\alpha\right\}\right), \quad n \rightarrow \infty. \quad (3)$$

Moreover, the power  $\alpha$  in (3) is optimal even for the class of stationary martingale differences: for each  $\alpha \in (0, 1)$ , there exists a stationary sequence of martingale differences  $(\hat{X}_i, \mathcal{F}_i)_{i \geq 1}$  satisfying  $E \exp\{|\hat{X}_1|^{\frac{2\alpha}{1-\alpha}}\} < \infty$  and

$$P\left(\max_{1 \leq k \leq n} \hat{S}_k \geq n\right) \geq \exp\{-3n^\alpha\}, \quad (4)$$

for all  $n$  large enough.

It is clear that when  $\alpha = \frac{1}{3}$ , the bound (3) implies the bound (1) of Lesigne and Volný.

In our second result we replace the condition  $\sup_i E \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$  of Theorem 2.1 by the weaker condition  $\sup_i E \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} < \infty$ , where  $X_i^+ = \max\{X_i, 0\}$ . Denote by

$$\langle S \rangle_k = \sum_{i=1}^k E(X_i^2 | \mathcal{F}_{i-1})$$

the sum of conditional variances.

**Theorem 2.2.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $\sup_i E \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$  for some constant  $C_1 \in (0, \infty)$ . Then, for all  $x, v > 0$ ,*

$$\begin{aligned} P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ \leq \exp\left\{-\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})}\right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (5)$$

We note that, for bounded random variables, some inequalities closely related to (5) can be found in Freedman [5], Dedecker [1], Dzharidze and van Zanten [3], Merlevède, Peligrad and Rio [12] and Delyon [2].

Adding a hypothesis on  $\langle S \rangle_n$  to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [11] for weakly dependent sequences.

**Corollary 2.1.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $E \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$  and  $E \exp\{(\frac{\langle S \rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\} \leq C_2$  for some constants  $C_1, C_2 \in (0, \infty)$ . Then, for all  $x > 0$ ,*

$$P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2\left(1 + \frac{1}{3}x\right)}n^\alpha\right\} + (nC_1 + C_2)\exp\{-x^\alpha n^\alpha\}. \quad (6)$$

In particular, with  $x = 1$ , it holds

$$P\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-Cn^\alpha\}), \quad n \rightarrow \infty, \quad (7)$$

where  $C > 0$  is an absolute constant. Moreover, the power  $\alpha$  in (7) is optimal even for the class of stationary martingale differences: for each  $\alpha \in (0, 1)$ , there exists a stationary sequence of martingale differences  $(\hat{X}_i, \mathcal{F}_i)_{i \geq 1}$  satisfying  $E \exp\{|\hat{X}_1|^{\frac{2\alpha}{1-\alpha}}\} < \infty$  and

$$P\left(\max_{1 \leq k \leq n} \hat{S}_k \geq n\right) \geq \exp\{-3n^\alpha\}, \quad (8)$$

for all  $n$  large enough.

In the i.i.d. case, the conditions of Corollary 2.1 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if  $E \exp\{(X_i^+)^{\alpha}\} < \infty$  with  $\alpha \in (0, 1)$ , then

$$P\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-C_\alpha n^\alpha\}), \quad n \rightarrow \infty. \quad (9)$$

### 3. Proof of Theorem 2.1

We need the following refined version of the Azuma-Hoeffding inequality.

**Lemma 3.1.** *Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of martingale differences satisfying  $|X_i| \leq 1$  for all  $i$ . Then, for all  $x \geq 0$ ,*

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2n}\right\}. \quad (10)$$

A proof can be found in Laib [7].

Now, we are ready to prove Theorem 2.1. We start as in Lesigne and Volný [9] and push a step further by using the martingale maximal inequality (10). We end by giving a simple example to show that the power  $\alpha$  in (3) is optimal.

Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of supermartingale differences. Given  $u > 0$ , define

$$\begin{aligned} X'_i &= X_i \mathbf{1}_{\{|X_i| \leq u\}} - E(X_i \mathbf{1}_{\{|X_i| \leq u\}} | \mathcal{F}_{i-1}), \\ X''_i &= X_i \mathbf{1}_{\{|X_i| > u\}} - E(X_i \mathbf{1}_{\{|X_i| > u\}} | \mathcal{F}_{i-1}), \\ S'_k &= \sum_{i=1}^k X'_i, \quad S''_k = \sum_{i=1}^k X''_i, \quad S'''_k = \sum_{i=1}^k E(X_i | \mathcal{F}_{i-1}). \end{aligned}$$

Then  $(X'_i, \mathcal{F}_i)_{i \geq 1}$  and  $(X''_i, \mathcal{F}_i)_{i \geq 1}$  are two martingale difference sequences and  $S_k = S'_k + S''_k + S'''_k$ . Let  $t \in (0, 1)$ . Since  $S'''_k \leq 0$ , for any  $x > 0$ ,

$$\begin{aligned} P\left(\max_{1 \leq k \leq n} S_k \geq x\right) &\leq P\left(\max_{1 \leq k \leq n} S'_k + S'''_k \geq xt\right) + P\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \\ &\leq P\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) + P\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right). \end{aligned} \quad (11)$$

Using Lemma 3.1 and  $|X'_i| \leq 2u$ , we have

$$P\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) \leq \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\}. \quad (12)$$

Let  $F_i(x) = P(|X_i| \geq x)$ ,  $x \geq 0$ . Since  $E \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$ , we obtain, for all  $x \geq 0$ ,

$$F_i(x) \leq \exp\{-x^{\frac{2\alpha}{1-\alpha}}\} E \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1 \exp\{-x^{\frac{2\alpha}{1-\alpha}}\}.$$

Using the martingale maximal inequality p. 14 in [6], we get

$$P\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^n EX_i''^2. \quad (13)$$

It is easy to see that

$$\begin{aligned} EX_i''^2 &= - \int_u^\infty t^2 dF_i(t) \\ &= u^2 F_i(u) + \int_u^\infty 2t F_i(t) dt \\ &\leq C_1 u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} + 2C_1 \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt. \end{aligned} \quad (14)$$

Notice that the function  $g(t) = t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\}$  is decreasing in  $[\beta, +\infty)$  and is increasing in  $[0, \beta]$ , where  $\beta = \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{2\alpha}}$ . If  $0 < u < \beta$ , we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &\leq \int_u^\beta t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\beta t \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} \beta^3 \exp\{-\beta^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \frac{3}{2} \beta^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (15)$$

If  $\beta \leq u$ , we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &= \int_u^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\infty t^{-2} u^3 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt \\ &= u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (16)$$

Returning to (14), by (15) and (16), we get

$$EX_i''^2 \leq 3C_1(u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (17)$$

From (13), it follows that

$$P\left(\max_{1 \leq k \leq n} S_k'' \geq x(1-t)\right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (18)$$

Combining (11), (12) and (18), we obtain

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq 2 \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\} + \frac{3nC_1}{(1-t)^2} \left(\frac{u^2}{x^2} + \frac{\beta^2}{x^2}\right) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.$$

Taking  $t = \frac{1}{\sqrt{2}}$  and  $u = \left(\frac{x}{4\sqrt{n}}\right)^{1-\alpha}$ , we get, for all  $x > 0$ ,

$$P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq C_n(\alpha, x) \exp\left\{-\left(\frac{x^2}{16n}\right)^\alpha\right\},$$

where

$$C_n(\alpha, x) = 2 + 35nC_1 \left(\frac{1}{x^{2\alpha}(16n)^{1-\alpha}} + \frac{\beta^2}{x^2}\right).$$

Hence, for all  $x > 0$ ,

$$P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq C(\alpha, x) \exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^\alpha\right\},$$

where

$$C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right).$$

This completes the first assertion of Theorem 2.1.

Next, we prove that the power  $\alpha$  in (3) is optimal. We take a positive random variable  $X$  such that, for all  $x > 1$ ,

$$P(X \geq x) = \frac{2e}{1 + x^{\frac{1+\alpha}{1-\alpha}}} \exp \left\{ -x^{\frac{2\alpha}{1-\alpha}} \right\}. \quad (19)$$

It is easy to verify that

$$E \exp \{ |X|^{\frac{2\alpha}{1-\alpha}} \} = - \int_1^\infty \exp \{ t^{\frac{2\alpha}{1-\alpha}} \} dP(X \geq t) = e + \frac{4e\alpha}{1-\alpha} \int_1^\infty \frac{t^{\frac{3\alpha-1}{1-\alpha}}}{1 + t^{\frac{1+\alpha}{1-\alpha}}} dt < \infty.$$

Assume that  $(\xi_i)_{i \geq 1}$  are Rademacher random variables independent of  $X$ , i.e.  $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$ . Set  $\hat{X}_i = X\xi_i$  and  $\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1, \dots, i})$ . Then,  $(\hat{X}_i, \mathcal{F}_i)_{i \geq 1}$  is a stationary sequence of martingale differences satisfying  $\sup_i E \exp \{ |\hat{X}_i|^{\frac{2\alpha}{1-\alpha}} \} = E \exp \{ |X|^{\frac{2\alpha}{1-\alpha}} \} < \infty$ . For  $\beta \in (0, 1)$ , it is easy to see that

$$P \left( \max_{1 \leq k \leq n} \hat{S}_i \geq n \right) \geq P \left( \hat{S}_n \geq n \right) \geq P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) P(X \geq n^{1-\beta}).$$

Since, for  $n$  large enough,

$$P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) \geq \exp \{ -n^{2\beta-1} \},$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for  $n$  large enough,

$$P \left( \max_{1 \leq k \leq n} \hat{S}_i \geq n \right) \geq \frac{2e}{1 + (n^{1-\beta})^{\frac{1+\alpha}{1-\alpha}}} \exp \left\{ -n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}} \right\}. \quad (20)$$

Setting  $2\beta - 1 = \alpha$ , we obtain, for  $n$  large enough,

$$P \left( \max_{1 \leq k \leq n} \hat{S}_i \geq n \right) \geq \frac{2e}{1 + n^{\frac{1+\alpha}{2}}} \exp \{ -2n^\alpha \} \geq \exp \{ -3n^\alpha \},$$

which proves that the power  $\alpha$  in (3) is optimal.

#### 4. Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality whose proof can be found in Fan, Grama and Liu [4].

**Lemma 4.1.** Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  are supermartingale differences satisfying  $X_i \leq 1$  for all  $i$ . Then, for all  $x, v > 0$ ,

$$P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x)} \right\}. \quad (21)$$

Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  are supermartingale differences. Given  $u > 0$ , set

$$X'_i = X_i \mathbf{1}_{\{X_i \leq u\}}, \quad X''_i = X_i \mathbf{1}_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^k X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^k X''_i.$$

Then,  $(X'_i, \mathcal{F}_i)_{i \geq 1}$  is also a sequence of supermartingale differences and  $S_k = S'_k + S''_k$ . Since  $\langle S' \rangle_k \leq \langle S \rangle_k$ , we deduce, for any  $x, u, v > 0$ ,

$$\begin{aligned} & P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq P(S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \quad + P(S''_k \geq 0 \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq P(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) + P\left(\max_{1 \leq k \leq n} S''_k \geq 0\right). \end{aligned} \quad (22)$$

Applying Lemma 4.1 to the supermartingale differences  $(X'_i/u, \mathcal{F}_i)_{i \geq 1}$ , we have

$$P(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\}. \quad (23)$$

Using the exponential Markov's inequality and the condition  $E \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$ , we get

$$\begin{aligned} P\left(\max_{1 \leq k \leq n} S''_k \geq 0\right) & \leq \sum_{i=1}^n P(X_i > u) \\ & \leq \sum_{i=1}^n E \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}} - u^{\frac{\alpha}{1-\alpha}}\} \\ & \leq nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (24)$$

Combining the inequalities (22), (23) and (24) together, we obtain, for all  $x, u, v > 0$ ,

$$\begin{aligned} & P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\} + nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (25)$$

Taking  $u = x^{1-\alpha}$ , we get, for all  $x, v > 0$ ,

$$\begin{aligned} & P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (26)$$

This completes the proof of Theorem 2.2.

## 5. Proof of Corollary 2.1.

To prove Corollary 2.1 we make use of Theorem 2.2. It is easy to see that

$$\begin{aligned}
P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq P\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n \leq nv^2\right) \\
&\quad + P\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n > nv^2\right) \\
&\leq P(S_k \geq nx \text{ and } \langle S \rangle_k \leq nv^2 \text{ for some } k \in [1, n]) \\
&\quad + P(\langle S \rangle_n > nv^2).
\end{aligned} \tag{27}$$

By Theorem 2.2, it follows that, for all  $x, v > 0$ ,

$$\begin{aligned}
P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq \exp\left\{-\frac{x^2}{2(n^{\alpha-1}v^2 + \frac{1}{3}x^{2-\alpha})}n^\alpha\right\} \\
&\quad + nC_1 \exp\{-x^\alpha n^\alpha\} + P(\langle S \rangle_n > nv^2),
\end{aligned}$$

Using the exponential Markov's inequality and the condition  $E \exp\{(\frac{\langle S \rangle_n}{n})^{\frac{\alpha}{1-\alpha}}\} \leq C_2$ , we get, for all  $v > 0$ ,

$$P(\langle S \rangle_n > nv^2) \leq E \exp\left\{\left(\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}} - v^{2\frac{\alpha}{1-\alpha}}\right)\right\} \leq C_2 \exp\{-v^{2\frac{\alpha}{1-\alpha}}\}.$$

Taking  $v = (nx)^{\frac{1-\alpha}{2}}$ , we obtain, for all  $x > 0$ ,

$$P\left(\max_{1 \leq k \leq n} X_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2(1 + \frac{1}{3}x)}n^\alpha\right\} + (nC_1 + C_2) \exp\{-x^\alpha n^\alpha\},$$

which gives inequality (6).

Next, we prove that the power  $\alpha$  in (7) is optimal even for the class of stationary martingale differences. Let  $X$  be the positive random variable defined in (19). Let  $\widehat{X}_i = X\xi_i$  and  $\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1, \dots, i})$ , where  $(\xi_i)_{i \geq 1}$  are Rademacher random variables independent of  $X$ . Note that  $\frac{\langle \widehat{S} \rangle_n}{n} = X^2$  satisfies the condition

$$E \exp\left\{\left(\frac{\langle \widehat{S} \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}}\right\} = E \exp\{|X|^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

Using the same argument as in the proof of Theorem 2.1, we obtain, for  $n$  large enough,

$$P\left(\max_{1 \leq k \leq n} \widehat{S}_k \geq n\right) \geq \exp\{-3n^\alpha\},$$

which shows that the power  $\alpha$  in (7) is optimal.



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